1D Sheet Beam Model for Intense Space-Charge:
Application to Debye Screening and the Distribution of Particle Oscillation Frequencies in a Thermal Equilibrium Beam*

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Background: Can 1D modeling be physical?

1D x-x' phase space much simpler to model collective beam evolution

Possible Issue: Solution of Poisson's equation for electric self-field of a charge in free space is radically different as function of dimension

$$\boxed{\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \qquad\qquad \mathbf{E} = -\frac{\partial \phi}{\partial \mathbf{x}}}$$

1D Sheet Charge (infinite range):

$$\rho = \Sigma \delta(x) \qquad \Longrightarrow \qquad \left| -\frac{\partial \phi}{\partial x} \right| = \frac{\Sigma}{2\epsilon_0} \sim \text{const}$$

2D Long Charged Rod(long range):

$$\rho = \lambda \frac{\delta(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}} \qquad \Longrightarrow \qquad \left| -\frac{\partial \phi}{\partial \mathbf{x}} \right| = \frac{\lambda}{2\pi\epsilon_0\sqrt{x^2+y^2}} \sim \frac{1}{\text{distance}}$$

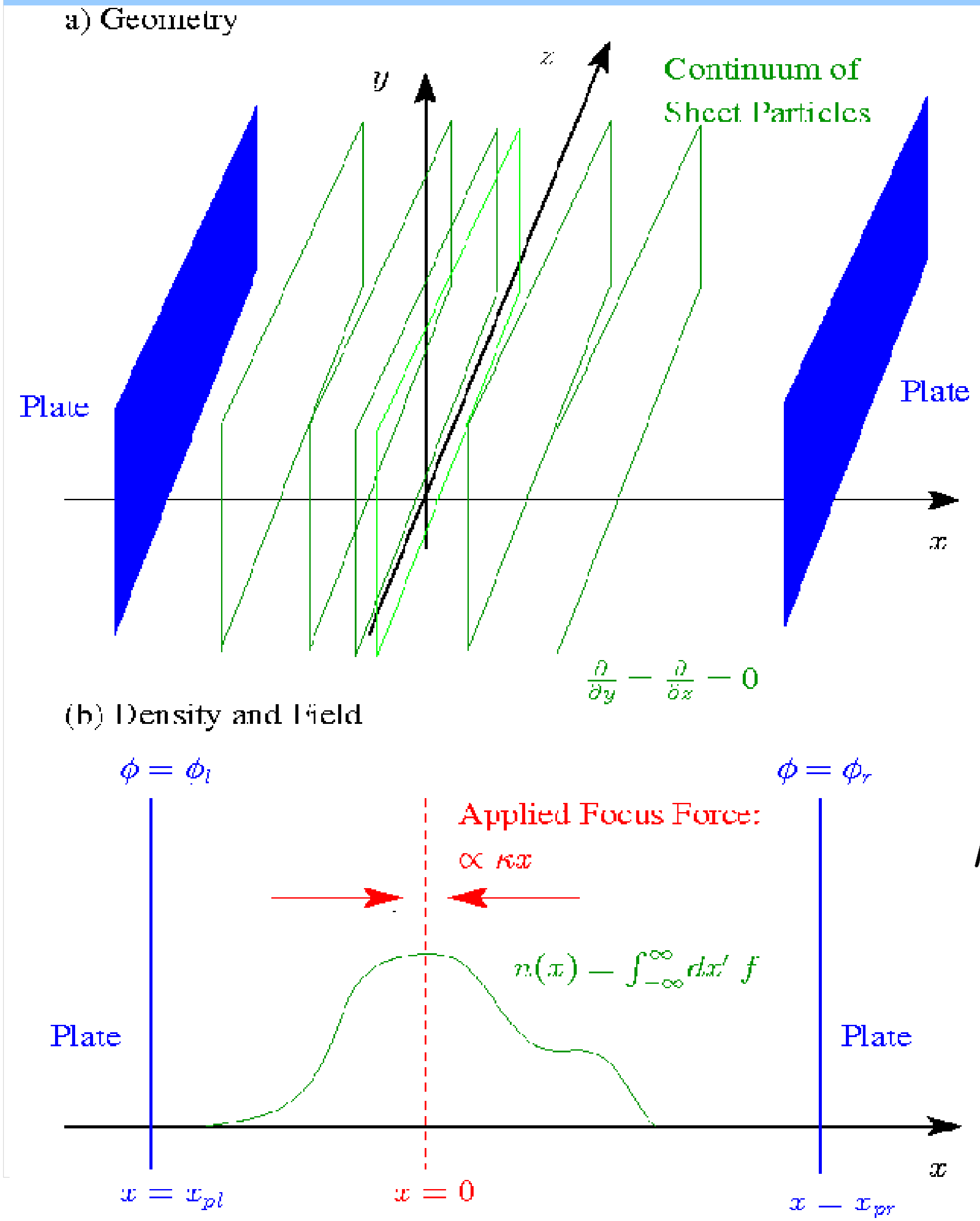
3D Point Charge (inverse square distance):

$$\rho = q\delta(x)\delta(y)\delta(z) \qquad \Longrightarrow \qquad \left| -\frac{\partial \phi}{\partial \mathbf{x}} \right| = \frac{q}{4\pi\epsilon_0(x^2+y^2+z^2)} \sim \frac{1}{(\text{distance})^2}$$

Can a beam model with 1D self-field produce physically relevant results?

- If so, adds relevance to interesting 1D collective mode results by Sacherer, Anderson, Okamoto, Startsev and Davidson, and others

1D Sheet Beam for intense beam modeling



Beam Uniform in *y*,*z*
Streaming along *z* with:

$$\beta_b c = \text{const}$$
$$= \text{Axial Velocity}$$

$$\gamma_b = 1/\sqrt{1-\beta_b^2}$$

$$\phi = \text{Electrostatic Potential}$$
$$\kappa(s) = \text{Focusing Function}$$

Relate to applied fields and phase advance σ_0 as usual

1D Vlasov-Poisson System

The sheet beam evolves in *x*-*x*' phase-space according to Vlasov's equation:

$$\boxed{\left\{ \frac{\partial}{\partial s} + \frac{\partial H}{\partial x'}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial x'} \right\} f(x, x', s) = 0}$$

with

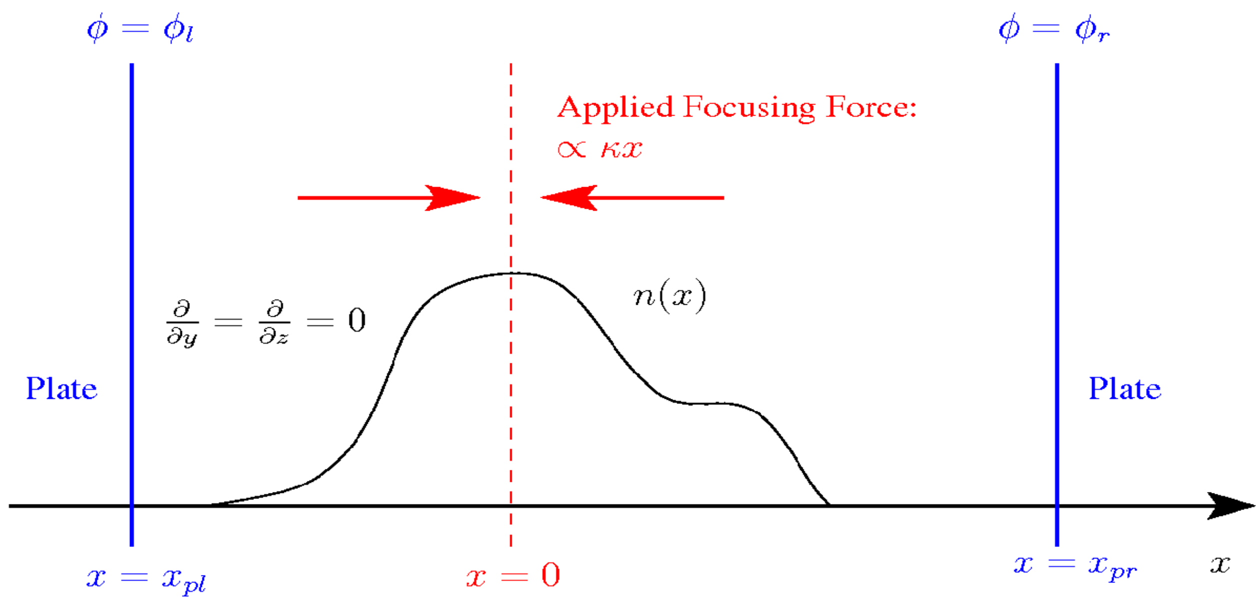
$$\boxed{H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^2}}$$

and coupling to the field specified by Poisson's equation

$$\frac{\partial^2}{\partial x^2}\phi = -\frac{q}{\epsilon_0} n \qquad\qquad n = \int_{-\infty}^{\infty} dx' \; f$$

+ Boundary Conditions

In 1D the Poisson equation simply solved for the field



$$\frac{\partial^2}{\partial x^2}\phi = -\frac{q}{\epsilon_0}n$$

$$n = \int_{-\infty}^{\infty} dx' \ f$$

Solution for electric field:

$$-\frac{\partial \phi}{\partial x} = -\frac{\phi_r - \phi_l}{x_{pr} - x_{pl}} - \frac{q}{\epsilon_0 (x_{pr} - x_{pl})} \int_{x_{pl}}^{x_{pr}} dx \ N_x \ - \frac{q N_x}{\epsilon_0}$$

$$N_x \equiv \int_{x_{pl}}^{x} d\tilde{x} \ n(\tilde{x}) \quad \propto \text{Charge to Left of } x$$

$$N \equiv \ N_x|_{x=x_{pr}} = \int_{x_{pl}}^{x_{pr}} d\tilde{x} \ n(\tilde{x}) \quad \propto \text{Total Charge}$$

(denote for later use)

Continuous focusing and beam stability

For continuous focusing:

$$\boxed{\kappa = k_{\beta 0}^2 = \text{const}}$$

the Hamiltonian is a constant of the motion

$$\boxed{H = \frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^2} = \text{const}}$$

and

$$\boxed{f(H) \geq 0 \quad \Longleftrightarrow \text{Equilibrium}}$$

For continuous focusing without bends, system conservation constraints:

"Probability":

$$U_G = \int_{x_{pl}}^{x_{pr}} dx \int_{-\infty}^{\infty} dx'\, G(f) \;= \text{const} \qquad \text{Any } G(f) \text{ with } G(f \to 0) = 0$$

Energy:

$$U_{\mathcal{E}} = \int_{x_{pl}}^{x_{pr}} dx \int_{-\infty}^{\infty} dx' \left\{\frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2\right\} f \;+ \int_{x_{pl}}^{x_{pr}} dx\, \frac{\epsilon_0 |\partial\phi/\partial x|^2}{2m\gamma_b^3\beta_b^2 c^2} \;= \text{const}$$

show a sufficient condition for beam stability is that

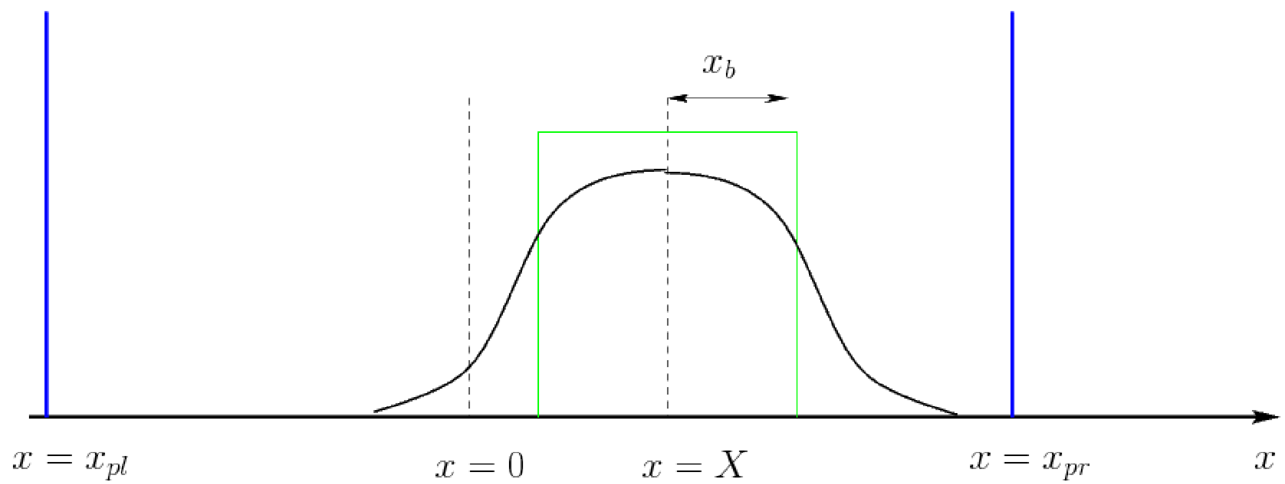
$$\boxed{df(H)/dH \leq 0 \quad \Longleftrightarrow \text{Stable Equilibrium}}$$

Centroid and Envelope Equations

Low order moment equations for 1st (centroid) and 2nd (envelope) order moments of distribution help interpret evolution

Statistical Average:

$$\langle \cdots \rangle = \frac{1}{N} \int_{x_{pl}}^{x_{pr}} dx \int_{-\infty}^{\infty} dx' \ \cdots f$$



Centroid Moments:

$$X = \langle x \rangle$$
$$X' = \langle x' \rangle$$

Envelope Moments:

$$x_b = \sqrt{3\langle (x-X)^2 \rangle}$$
$$x_b' = \frac{\sqrt{3}\langle (x-X)(x'-X') \rangle}{\sqrt{\langle (x-X)^2 \rangle}}$$

- Coefficient different than familiar 2D theory

Centroid Equation of Motion:

$$X'' + \kappa X = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\left[\left.\frac{\partial\phi}{\partial x}\right|_a + \left.\frac{\partial\phi}{\partial x}\right|_i\right]$$

$$= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\left[\frac{\phi_r - \phi_l}{x_{pl} - x_{pr}} + \frac{q}{\epsilon_0(x_{pl} - x_{pr})}\int_{x_{pl}}^{x_{pr}} dx\; N_x - \frac{qN}{2\epsilon_0}\right]$$

Applied (Bending) Term

Image Terms

Envelope Equation of Motion:

$$x_b'' + \kappa x_b - P\frac{3\left[\int_{x_{pl}}^{x_{pr}} dx\;\left(\frac{N_x}{N}\right) - \int_{x_{pl}}^{x_{pr}} dx\;\left(\frac{N_x}{N}\right)^2\right]}{x_b} - \frac{\varepsilon^2}{x_b^3} = 0$$

$$P \equiv \frac{q^2N}{2\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const}$$

1D sheet beam perveance

- $[P]$ = 1/length, contrast to 2D

$$\varepsilon \equiv 3\left[\langle\tilde{x}^2\rangle\langle\tilde{x}'^2\rangle - \langle\tilde{x}\tilde{x}'\rangle^2\right]^{1/2} \neq \text{const}$$

$$\tilde{x} = x - X$$

1D x-x' rms edge emittance

- Coefficient different than 2D theory
- Generally evolves

$$3\left[\int_{x_{pl}}^{x_{pr}} dx\;\left(\frac{N_x}{N}\right) - \int_{x_{pl}}^{x_{pr}} dx\;\left(\frac{N_x}{N}\right)^2\right] \neq \text{const}$$

Dimensionless Factor

- Generally evolves

RMS Equivalent Beam and KV Distribution

For a uniform density beam (density uniform within $x = X \pm x_b$ edge):

$$\boxed{n(x) = \int_{-\infty}^{\infty} dx'\ f \ = \begin{cases} 0, & X + x_b < x < x_{pr} \\ \hat{n}, & X - x_b < x < X + x_b \\ 0, & x_{pl} < x < X - x_b \end{cases}}$$

Reduced Centroid Equation of Motion:

Uniform density beam: $\Longrightarrow$

$$-\frac{q}{m\gamma_b^3\beta_b^2c^2}\left[\frac{q}{\epsilon_0(x_{pl}-x_{pr})}\int_{x_{pl}}^{x_{pr}} dx\ N_x - \frac{qN}{2\epsilon_0}\right] = \frac{2P}{x_{pr}-x_{pl}}\left(X - \frac{x_{pr}+x_{pl}}{2}\right)$$

$$\boxed{X'' + \kappa X = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\phi_r - \phi_l}{x_{pr}-x_{pl}} + \frac{2P}{x_{pr}-x_{pl}}\left(X - \frac{x_{pr}+x_{pl}}{2}\right)}$$

Reduced Moment Equation of Motion:

Uniform density beam: $\Longrightarrow$

$$\varepsilon = \text{const}$$

$$3\left[\int_{x_{pl}}^{x_{pr}} dx\ \left(\frac{N_x}{N}\right) - \int_{x_{pl}}^{x_{pr}} dx\ \left(\frac{N_x}{N}\right)^2\right] = x_b$$

$$\boxed{x_b'' + \kappa x_b - P - \frac{\varepsilon^2}{x_b^3} = 0}$$

P = const, dimension: $[P]$ = 1/length

Centroid and envelope equations of motion are decoupled and closed!

Self-consistent KV distribution generates the uniform density beam

$$f = \frac{N}{2\pi\varepsilon\sqrt{1-\left(\frac{\tilde{x}}{x_b}\right)^2-\left(\frac{x_b\tilde{x}'-x_b'\tilde{x}}{\varepsilon}\right)^2}}\Theta\left[1-\left(\frac{\tilde{x}}{x_b}\right)^2-\left(\frac{x_b\tilde{x}'-x_b'\tilde{x}}{\varepsilon}\right)^2\right]$$

$$\tilde{x} = x - X \qquad \tilde{x}' = x' - X'$$

- F. Sacherer, Ph.D. Thesis, University of California (1968)
- 1D sheet beam form very different than more familiar 2D KV distribution
  - Less singular (1/sqrt divergence rather than delta function divergence)
- Consistent with (linear force) image charges for 1D sheet beam!

An rms equivalent KV beam can then be defined for *any* distribution *f*:

Quantity	RMS Equivalent	Calculated From Distribution
Perveance	P	$= q^2N/(2\epsilon_0 m\gamma_b^3\beta_b^2c^2)$
Centroid Coordinate	X	$= \langle x\rangle$
Centroid Angle	X'	$= \langle x'\rangle$
Envelope Coordinate	x_b	$= \sqrt{3\langle\tilde{x}^2\rangle}$
Envelope Angle	x_b'	$= \sqrt{3}\langle\tilde{x}\tilde{x}'\rangle/\sqrt{\langle\tilde{x}^2\rangle}$
Emittance	ε	$= 3\sqrt{\langle\tilde{x}^2\rangle\langle\tilde{x}'^2\rangle - \langle\tilde{x}\tilde{x}'\rangle^2}$

For an rms equivalent KV beam with a matched envelope in a periodic lattice

$$\boxed{x_b(s + L_p) = x_b(s)}$$

a particle moving within the equivalent beam has phase advance:

$$\boxed{\sigma = \varepsilon \int_{s_i}^{s_i + L_p} \frac{ds}{x_b^2}}$$

Allows a convenient, normalized measure of space-charge strength:

Tune Depression:

$$\frac{\sigma}{\sigma_0} \in (0, 1)$$

$$\frac{\sigma}{\sigma_0} \rightarrow 1$$

Warm Beam,

Min space-charge

Particle moving in applied focus

$$\frac{\sigma}{\sigma_0} \rightarrow 0$$

Cold beam,

Max space-charge

Space-charge cancels applied focus

For continuous focusing, the tune depression can be simply expressed:

$$\boxed{\frac{\sigma}{\sigma_0} = \sqrt{1 - \frac{P}{k_{\beta 0}^2 x_b}} \qquad \kappa = k_{\beta 0}^2 = \text{const}}$$

Parametric Equivalence with 2D Beams

Equivalences relating more common 2D and 1D sheet beam parameters can be developed:

Applied Focus:

$$\boxed{\kappa(s) = \kappa_j(s)}$$

$j = x$ or y for 2D system

Perveance: (for same characteristic plasma frequency)

$$\boxed{P = \frac{2Q}{r_b}}$$

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2} = \text{ Usual 2D perveance}$$

$r_b =$ Characteristic 2D beam radius

Emittance:

$$\boxed{\varepsilon = \varepsilon_j}$$

$\varepsilon_j =$ usual x or y 2D rms edge emittance

$$\varepsilon_x = 4\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp\right]^{1/2}$$

For continuous focusing these results give:

$$\boxed{P = 2^{3/2} k_{\beta 0} Q \Big/ \sqrt{Q + \sqrt{Q^2 + 4k_{\beta 0}^2 \varepsilon_x^2}} \qquad \begin{aligned} \kappa &= k_{\beta 0}^2 = \text{const} \\ \varepsilon_x &= \varepsilon_y \end{aligned}}$$

Using these equivalences:

- Single particle orbits same as in higher dimensional models
- Centroid has correct single particle phase advance
  - Image scaling right sense in linear approx but modified form
- Envelope mode analysis shows:

  Only "breathing" symmetry envelop oscillation, but 1D mode frequency corresponds to 2D "quadrupole" mode oscillation

$$\sim e^{iks} \text{ mode variation}$$

$$\Longrightarrow \frac{k}{k_{\beta 0}} = \pm\sqrt{1 + 3\left(\frac{\sigma}{\sigma_0}\right)^2}$$

  Suggests 1D model may not be good for halo extent estimates

- Formulas relating emittance variation to excess field energy can be Derived in similar form to higher dimensional models (Wangler, Lapostolle)

Continuous Focusing: The Thermal Equilibrium Sheet Beam Distribution:

[2D: Davidson, *Physics of Nonneutral Plasmas* (1990); Reiser, *Theory and Design of Charged Particle Beams* (1994); PRSTAB **12**, 114801(2009), ...]

In a long continuous focusing channel with $\kappa = k_{\beta 0}^2 = \text{const}$, collisions eventually relax the beam to thermal equilibrium. The Fokker-Planck equation predicts the unique Maxwell-Boltzmann distribution describing this limit:

$$\lim_{s \to \infty} f \propto \exp\left(-\frac{H_{\text{rest}}}{T}\right)$$

$H_{\text{rest}} =$ single particle Hamiltonian of beam in rest frame (energy units)

$T = \text{const}$ Thermodynamic temperature (energy units)

Beam propagation time in transport channel is generally short relative to collision time, inhibiting full relaxation

- Collective effects may enhance relaxation rate
  - Wave spectra likely large for real beams and enhanced by transient and nonequilibrium effects
  - Random errors acting on system may enhance and lock-in phase mixing

Continuous focusing thermal equilibrium distribution

Analysis of the rest frame transformation shows the 1D Maxwell-Boltzmann distribution is:

$$\boxed{f(H) = \left(\frac{m\gamma_b\beta_b^2c^2}{2\pi T}\right)^{1/2} \hat{n}\exp\left(\frac{-m\gamma_b\beta_b^2c^2 H}{T}\right)}$$

$$H = \frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$T = \text{const}$ Temperature (energy units, lab frame)

$n(r=0) = \hat{n} = \text{const}$ on-axis density

$\phi(r=0) = 0$ (reference choice)

The density can be calculated in terms of the equilibrium potential ϕ as:

$$\boxed{n \equiv \int_{-\infty}^{\infty} dx'\ f \ = \hat{n}\exp\left[-\frac{m\gamma_b\beta_b^2c^2}{T}\left(\frac{1}{2}k_{\beta 0}^2x^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}\right)\right]}$$

and the kinetic temperature is spatially uniform with:

$$\boxed{T_x \equiv m\gamma_b\beta_b^2c^2\frac{\int_{-\infty}^{\infty} dx'\ x'^2 f}{\int_{-\infty}^{\infty} dx'\ f} \ = T = \text{const}}$$

Analysis of system obtains a 1D nonlinear Poisson equation that is analogous to 2D equation

- Solve numerically or (approximately) analytically analogously to POP 15, 043101 (2008)

Interpret results in terms of rms equivalent beam tune depression:

$$\frac{\sigma}{\sigma_0} = \left[1 - \frac{P}{\sqrt{3}k_{\beta 0}^2 \sqrt{\langle x^2 \rangle}}\right]^{1/2}$$

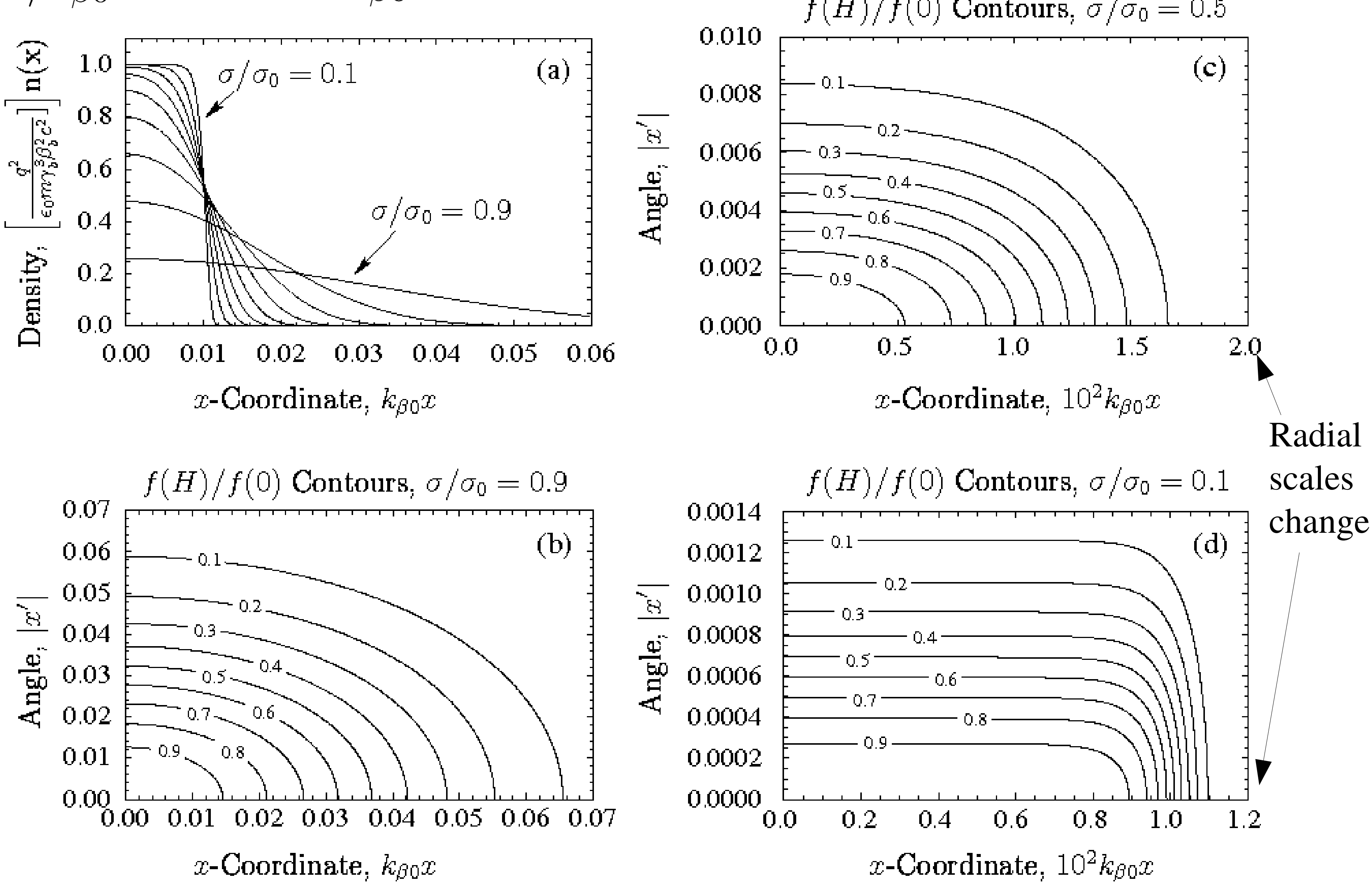
$$\frac{\sigma}{\sigma_0} \in [0, 1]$$

$\sigma/\sigma_0 \to 1$	Warm beam, Min space-charge	$\Longleftrightarrow \hat{n} \to 0$
$\sigma/\sigma_0 \to 0$	Cold beam, Max space-charge	$\Longleftrightarrow T \to 0$

Constraints derived/solved to hold relevant parameters fixed and illustrate equilibrium characteristics at fixed pervance P, focusing strength $k_{\beta 0}$ as a function of σ/σ_0

Distribution contours at fixed charge and focusing strength

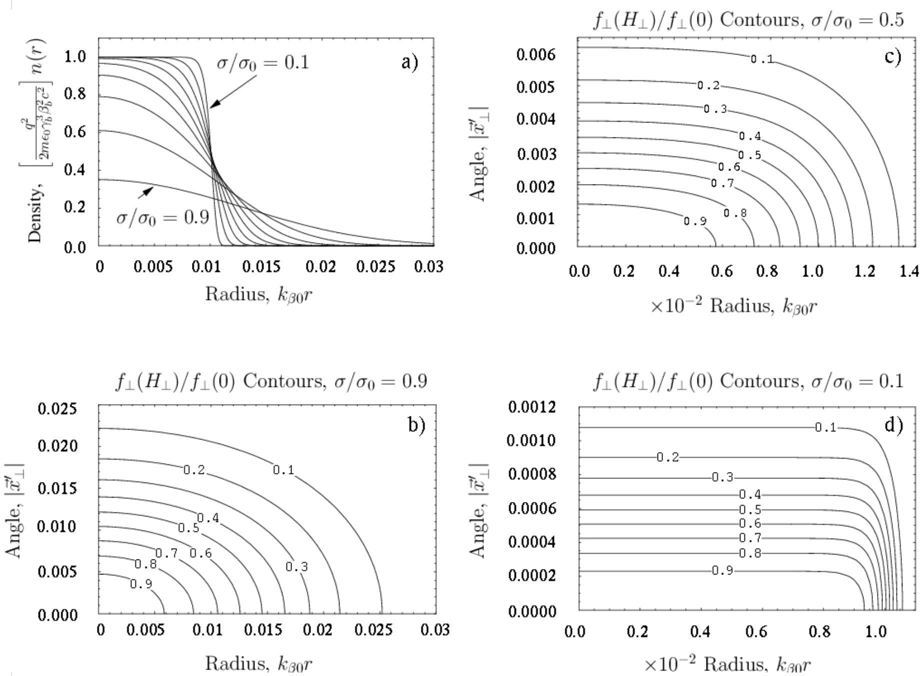
$P/k_{\beta 0} = 0.02 \qquad k_{\beta 0}^2 = \text{const}$



- For strong space-charge particles move approx force-free in core till approaching the edge where it is rapidly (nonlinearly) reflected

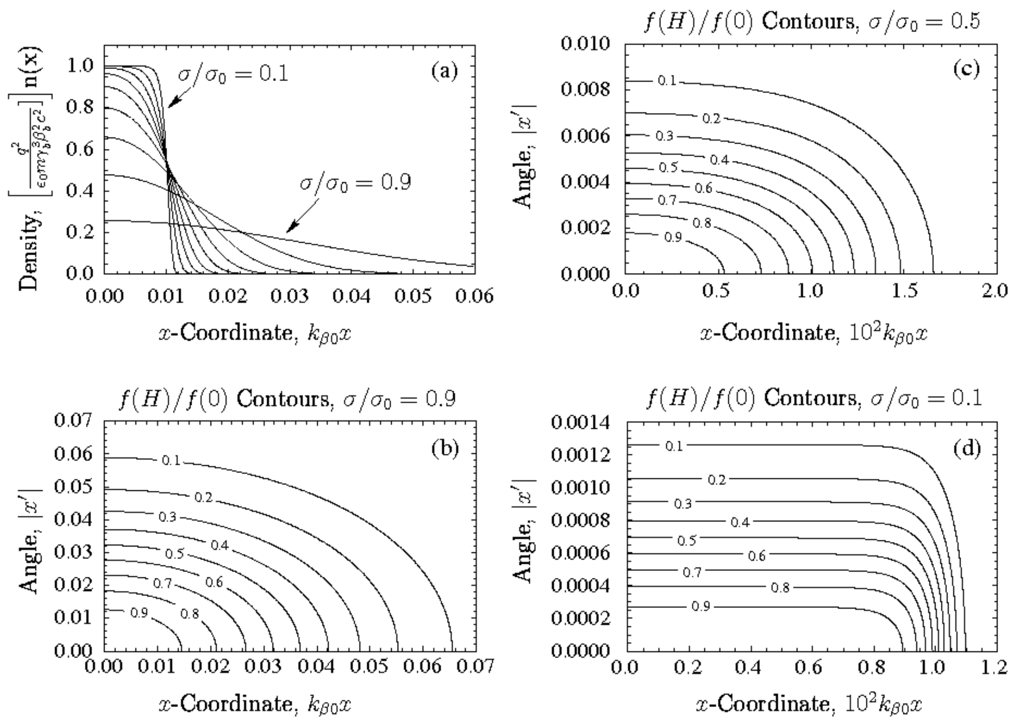
Phase-space properties of the distributions in 2D and 1D are very similar in spite of Coulomb force being radically different in 2D and 1D

2D Cylindrical Beam



PRSTAB **12**, 114801(2009)

1D Sheet Beam



How are 2D and 1D results so similar?

- Debye screening of linear applied focus force partly explains

Debye Screening in a Thermal Equilibrium Sheet Beam

Space-charge and the applied focusing forces of the lattice work together to Debye screen interactions in the core of a beam with high space-charge intensity

Review:

Free-space field of a "bare" test sheet-charge Σ_t at the origin $x = 0$

$$\rho(r) = \Sigma_t \delta(x) \qquad\qquad \frac{\partial^2}{\partial x^2}\phi = -\frac{\rho}{\epsilon_0} = -\frac{\Sigma_t}{\epsilon_0}\delta(x)$$

solution shows long-range interaction

$$\boxed{-\frac{\partial \phi}{\partial x} = \mathrm{sgn}(x)\frac{\Sigma_t}{2\epsilon_0}}$$

Follow analysis in Davidson, *Physics of Nonneutral Plasmas* (1990), set:

$$\phi = \phi_0 + \delta\phi \qquad \begin{array}{rl} \phi_0 = & \text{Thermal Equilibrium potential with no test sheet-charge} \\ \delta\phi = & \text{Perturbed potential from test sheet-charge} \end{array}$$

Place a *small* test sheet charge at $x = 0$ in a thermal equilibrium beam and assume:

- Equilibrium adiabatically adapts to test charge
- Equilibrium relatively cold so density profile is flat

This gives:

$$\frac{\partial^2}{\partial x^2}\delta\phi - \frac{\delta\phi}{(\gamma_b\lambda_D)^2} \simeq -\frac{\Sigma_t}{\epsilon_0}\delta(x)$$

$$\lambda_D = \left(\frac{\epsilon_0 T}{q^2\hat{n}}\right)^{1/2} = $$ Debye radius formed from peak, on-axis beam density

Derive a general solution by connecting solution very near the test sheet-charge with the general solution for *x* nonzero:

Potential: $$\delta\phi(x) \simeq \frac{\gamma_b\lambda_D\Sigma_t}{2\epsilon_0}e^{-|x|/(\gamma_b\lambda_D)}$$

Field: $$-\frac{\partial\delta\phi}{\partial x} \simeq \mathrm{sgn}(x)\frac{\Sigma_t}{2\epsilon_0}e^{-|x|/(\gamma_b\lambda_D)}$$

Classic Debye screened interaction form

- Applied focus force takes role of 2^{nd} (stationary) neutralizing species
- Beam particles redistribute to screen bare interaction
- Expect beam to behave as a plasma with similar collective waves etc.

Compare result to higher dimensional models of thermal equilibrium beams:

Dimension	Distance Measure	Test Charge Density $\rho =$	Screened Potential $\delta\phi \simeq$
1D	$\|x\|$	$\Sigma_t \delta(x)$	$\frac{\gamma_b \lambda_D \Sigma_t}{2\epsilon_0} e^{-\|x\|/(\gamma_b \lambda_D)}$
2D	$r = \sqrt{x^2 + y^2}$	$\lambda_t \frac{\delta(r)}{2\pi r}$	$\frac{\lambda_t}{2\sqrt{2\pi}\epsilon_0} \frac{r}{\sqrt{r/(\gamma_b \lambda_D)}} e^{-r/(\gamma_b \lambda_D)}$ $r \gg \gamma_b \lambda_D$
3D	$r = \sqrt{x^2 + y^2 + z^2}$	$q_t \delta(x)\delta(y)\delta(z)$	$\frac{q_t}{4\pi\epsilon_0 r} e^{-r/(\gamma_b \lambda_D)}$

- Essentially same result in 1D, 2D, and 3D
  - Expect similar collective effects in 1D, 2D, and 3D
  - Reason why lower dimension models can get the "right" answer for collective interactions in spite of the Coulomb force varying with dimension
- Explains why the radial density profile in the core of space-charge dominated beams are expected to be flat for linear applied focusing forces
  - Linear charge => Linear self-field force to cancel linear applied force
  - Expected to happen for any reasonable smooth distribution
    See examples in: PRSTAB **12**, 114801(2009)

Distribution of particle oscillation frequencies in a continuously focused Thermal Equilibrium Sheet Beam

Nonlinear oscillation wavelength λ of a particle with Hamiltonian H

$$\lambda = \oint_{\text{orbit}} ds \;=\; 2^{3/2} \int_0^{x_t} \frac{dx}{\sqrt{H - \left(\frac{1}{2} k_{\beta 0}^2 x^2 + \frac{q\phi}{m\gamma_b^3 \beta_b^2 c^2}\right)}}$$

Turning point: $\dfrac{1}{2} k_{\beta 0}^2 x_t^2 - \dfrac{q\phi(x = x_t)}{m\gamma_b^3 \beta_b^2 c^2} = H$

For a thermal equilibrium, apply probability transform to calculate frequency distribution in normalized form:

$$\frac{k_\beta}{k_{\beta 0}} = \frac{\lambda_0}{\lambda} = \frac{2\pi}{(k_{\beta 0}\lambda)} \quad \in [0\ ,1]$$

$\lambda_0 =$ Wavelength in Applied Focus

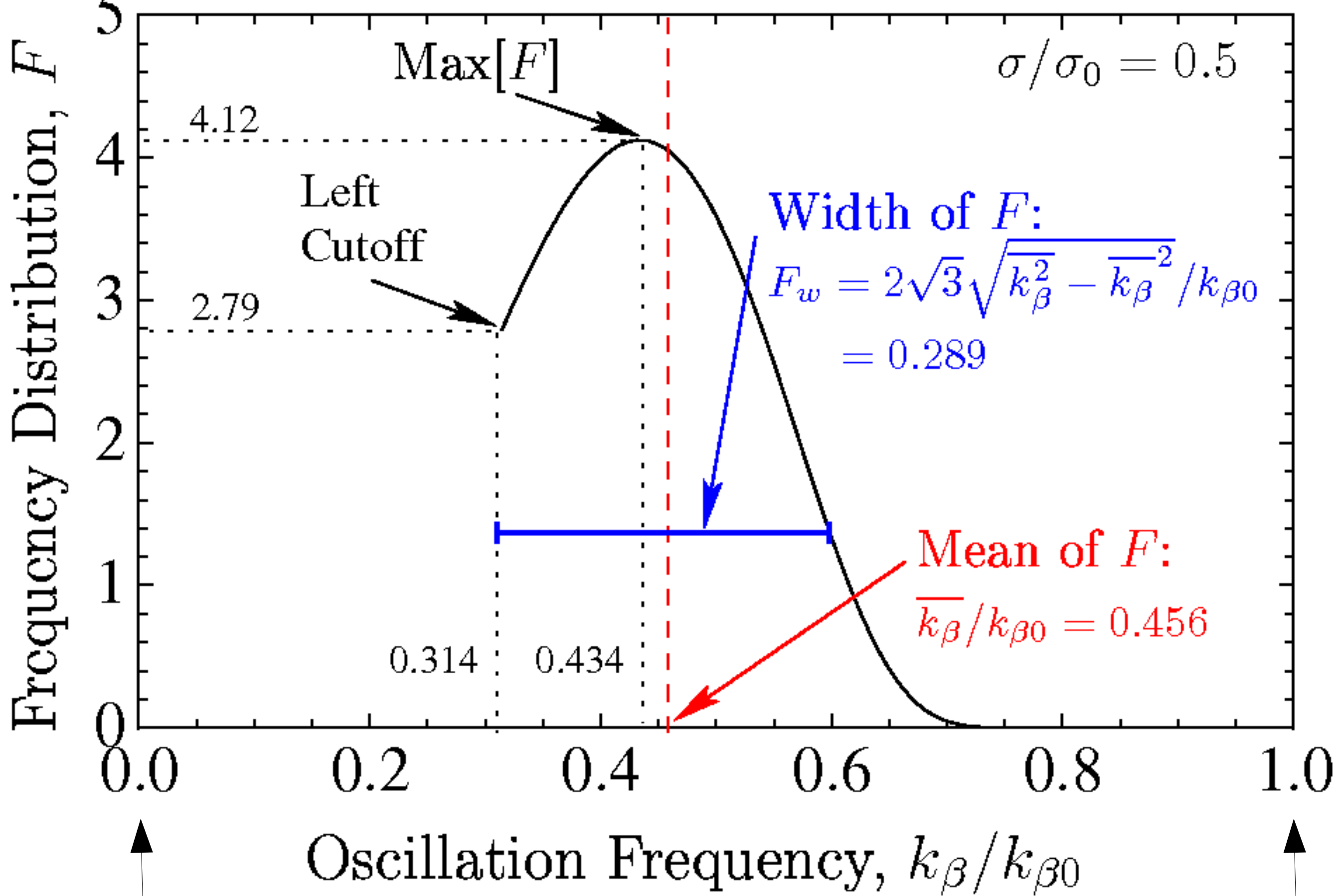
$\dfrac{k_\beta}{k_{\beta 0}} \to 1$ Limit of zero space-charge intensity

Particle moving in applied focus $(\lambda = \lambda_0)$

$\dfrac{k_\beta}{k_{\beta 0}} \to 0$ Limit of max space-charge intensity

Applied focus force canceled $(\lambda \to \infty)$

Apply procedure for a single value of rms equivalent beam σ/σ_0



Min: Full Depression

Max: Applied Focus

Mean: $\mu_F \equiv \overline{k_\beta}/k_{\beta 0}$

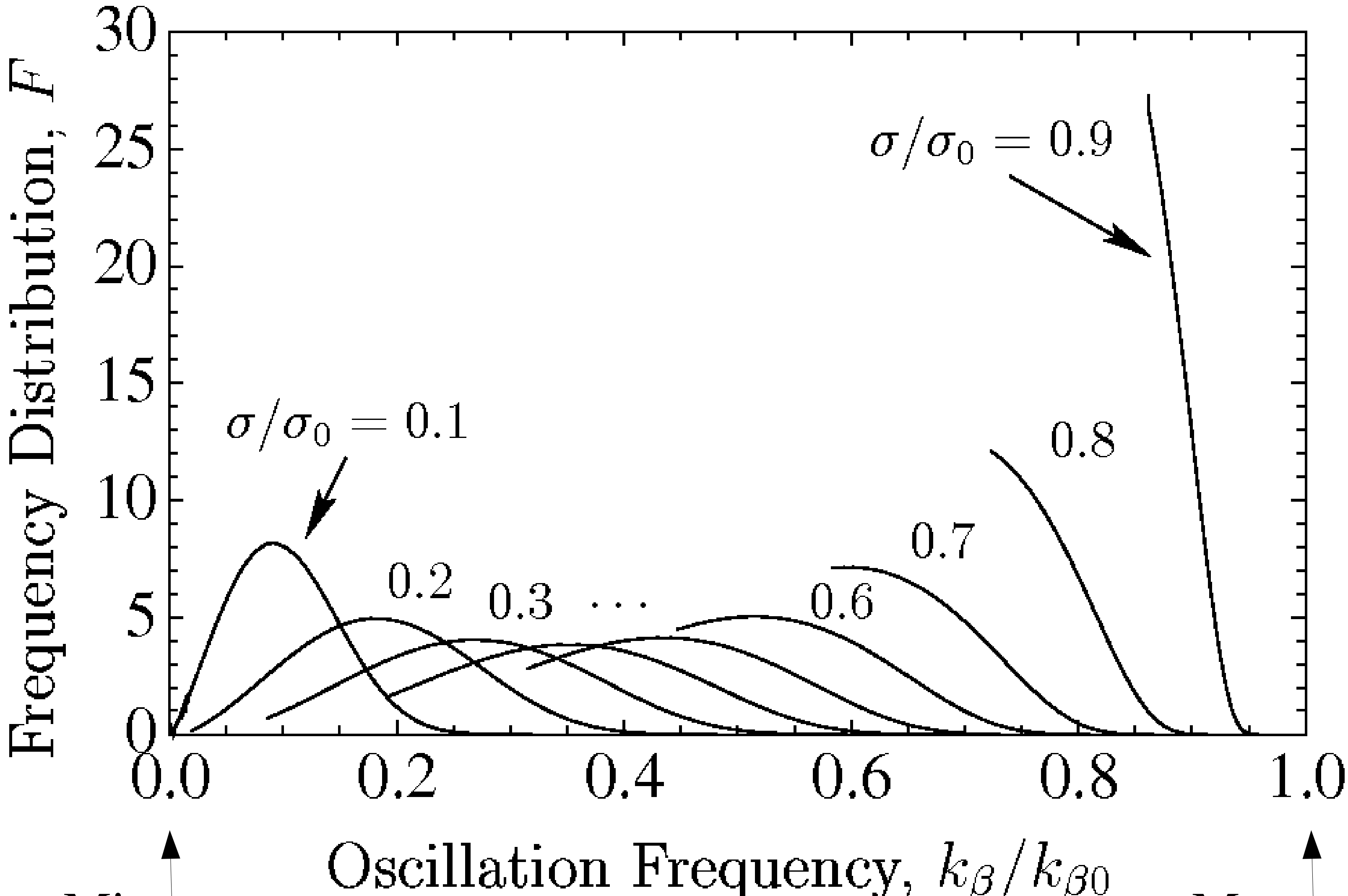
RMS: $\sigma_F \equiv \sqrt{\overline{(k_\beta - \overline{k_\beta})^2}}/k_{\beta 0} = \sqrt{\overline{k_\beta^2} - \overline{k_\beta}^2}/k_{\beta 0}$

Width: $F_w \equiv 2\sqrt{3}\sigma_F$

Relative Width: F_w/μ_F

$$\overline{\cdots} = \int_0^1 d(k_\beta/k_{\beta 0}) \; \cdots F$$

Superimposed results for rms equivalent beam values of σ/σ_0 show how the distribution of oscillator frequencies in a thermal equilibrium sheet beam changes as space charge intensity is varied



Min:
Full Depression

Max:
Applied Focus

Discussion points:

- Most features of thermal equilibrium results should roughly apply to any choice of smooth equilibrium distribution. For strong space-charge expect:
  - Broad distribution of particle oscillation frequencies
  - Large range of oscillation amplitude moves nearly force free in core due to Debye screening till being nonlinearly reflected in the beam edge
- Broad frequency distribution suggests robust stability properties:
  - All modes stable for thermal equilibrium
  - To the extent can extrapolate results to non-equilibrium distributions in periodic focusing channels helps explain the robust beam stability observed in experiment and simulations for very strong space charge:

    Tiefenback, Ph.d. Thesis, University of California at Berkeley (1986)

    NIMA **561** 203 (2006); NIMA **577** 173 (2007)
- Rms equivalent KV beam does not accurately model the average frequency in the distribution except for weak space charge.
- Suggests odd feature of KV model applied to space-charge mode resonances:
  - For weak space charge with $\sigma/\sigma_0 \lesssim 1$, KV model should work adequately:

    Freq dist highly peaked about avg space-charge shifted value in spite of nonuniform charge distribution
  - For strong space-charge with $\sigma/\sigma_0 \lesssim 0.75$, KV model poor:

    Freq dist very broad in spite of increasingly uniform charge distribution

Conclusions

- Sheet beam model developed for simplified analysis of beams with intense space-charge
  - 1D structure simplifies analysis
    - Poisson equation for self-field simple form but long range
  - Image charges possible to model
  - Rms equivalent beam and envelope equation analogous to 2D case
- Simple 1D sheet beam model used to illuminate several features with a continuously focused thermal equilibrium beam

  1) Equilibrium very similar to 2D systems suggesting good model

  2) Debye screening same in 1D as 3D and 2D systems
     - Suggests collective effects closely model higher dimensional systems in spite of the very different Coulomb force

  3) Frequency distribution calculated
     - Space-charge strongly broadens distribution
       suggesting robust stability for high space-charge with smooth distributions

Published article details work:

Lund, Friedman, and Bazouin, PRSTAB 14, 054201 (2011)

Work on periodic focusing (simulations; M. Campos-Pinto) to be submitted